KEY

Math 343 Midterm 3
Instructor: Scott Glasgow
Sections: 1, 6 and 8
Dates: November 17th and 18th, 2005

Instructions: As usual your work that I ultimately see and grade should be a (logical) work of art. More importantly realize that despite the length of the verbage in communicating this exam, most of this exam is *instructional* in nature, which, among other things, means that I often give you the answer (perhaps slyly), and hope only that a) you will show why the obvious answer holds and that b) you will muse on such facts to get a better and bigger picture of the relationships between various recent topics. Thus I expect many perfect scores. But if you find yourself getting confused, sit back, relax, and realize that on the typical problem there is rather little to do (besides the occasional and tedious row-reduction or matrix multiplication—for which your simple calculator should come in handy) . At any rate, for each question, try to ask yourself "What is being asked?" and distinguish that from what is supposed to be helpful explanations of the mentioned connections. If you do not find my explanations of connections between topics helpful, then by all means **ignore them**. Finally, as usual, USE YOUR OWN PAPER.

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Note: to my knowledge all problems are "well-engineered" to have integer solutions. If you do not get integer solutions (when you have a choice—you may not when "normalization" is required), try again. (But even when normalization is required, I have made the vectors only as bad as rational—not irrational. This requires some doing!)

1. Prove that if A and B are orthogonal matrices, then

i. A^{-1} is an orthogonal matrix

ii. AB is an orthogonal matrix

iii. $\det A = \pm 1$.

10 pt.s

Solution

i. A is an orthogonal matrix $\Leftrightarrow A^{-1} = A^T \Leftrightarrow \left(A^{-1}\right)^{-1} = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T \Leftrightarrow A^{-1}$ is an orthogonal matrix.

iv. A and B are orthogonal matrices
$\Leftrightarrow A^{-1} = A^T$ and $B^{-1} = B^T \Rightarrow \left(AB\right)^{-1} = B^{-1}A^{-1} = B^T A^T = \left(AB\right)^T \Leftrightarrow AB$ is an orthogonal matrix.

ii. A is an orthogonal matrix
$\Leftrightarrow A^{-1} = A^T \Rightarrow 1 = \det I = \det A^T A = \det A^T \det A = \left(\det A\right)^2 \Leftrightarrow \det A = \pm 1$.

2. Given

$\mathbf{q}_1(x) := 1,\ \mathbf{q}_2(x) := \sqrt{3}\ \left(2x-1\right),\ \mathbf{q}_3(x) := \sqrt{5}\ \left(6x^2-6x+1\right),\ \mathbf{b}(x) := 20x^3$

$\mathbf{b}_\perp(x) := 20x^3 - 30x^2 + 12x - 1,\ \mathbf{f}\cdot\mathbf{g} := \left\langle \mathbf{f},\mathbf{g}\right\rangle := \int_0^1 \mathbf{f}(x)\mathbf{g}(x)dx, \left\|\mathbf{f}\right\| := \left\langle \mathbf{f},\mathbf{f}\right\rangle^{1/2}$, and

percent relative error $:= 100\left\|\mathbf{b}_\perp\right\| / \left\|\mathbf{b}\right\|$,

compute the following quantities:

i. $\mathbf{q}_1 \cdot \mathbf{q}_2$

ii. $\mathbf{q}_1 \cdot \mathbf{q}_3$

iii. $\mathbf{q}_1 \cdot \mathbf{b}_\perp$

iv. $\mathbf{q}_2 \cdot \mathbf{q}_3$

v. $\mathbf{q}_2 \cdot \mathbf{b}_\perp$

vi. $\mathbf{q}_3 \cdot \mathbf{b}_\perp$

vii. $(\mathbf{b} - \mathbf{b}_\perp) \cdot \mathbf{b}_\perp$

viii. $(\mathbf{q}_1 \cdot \mathbf{b})\mathbf{q}_1(x) + (\mathbf{q}_2 \cdot \mathbf{b})\mathbf{q}_2(x) + (\mathbf{q}_3 \cdot \mathbf{b})\mathbf{q}_3(x) + \mathbf{b}_\perp(x) - \mathbf{b}(x)$

ix. percent relative error -5.

Important: please use the following Table of Integrals—

$\mathbf{g}$ \ $\mathbf{f}$: $\int_0^1 \mathbf{fg}(x)dx$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{b}_\perp$	$\mathbf{b}-\mathbf{b}_\perp$	$\mathbf{b}$
$\mathbf{q}_1$	1	0	0	0	5	5
$\mathbf{q}_2$	0	1	0	0	$3\sqrt{3}$	$3\sqrt{3}$
$\mathbf{q}_3$	0	0	1	0	$\sqrt{5}$	$\sqrt{5}$
$\mathbf{b}_\perp$	0	0	0	$1/7$	0	$1/7$
$\mathbf{b}-\mathbf{b}_\perp$	5	$3\sqrt{3}$	$\sqrt{5}$	0	57	57
$\mathbf{b}$	5	$3\sqrt{3}$	$\sqrt{5}$	$1/7$	57	$400/7$

(0.1)

15 pt.s

<u>Solution</u>

From table (0.1) we have

i. $\mathbf{q}_1 \cdot \mathbf{q}_2 = \int_0^1 \mathbf{q}_1\mathbf{q}_2(x)dx = 0$

ii. $\mathbf{q}_1 \cdot \mathbf{q}_3 = \int_0^1 \mathbf{q}_1\mathbf{q}_3(x)dx = 0$

iii. $\mathbf{q}_1 \cdot \mathbf{b}_\perp = \int_0^1 \mathbf{q}_1\mathbf{b}_\perp(x)dx = 0$

iv. $\mathbf{q}_2 \cdot \mathbf{q}_3 = \int_0^1 \mathbf{q}_2\mathbf{q}_3(x)dx = 0$

v. $\mathbf{q}_2 \cdot \mathbf{b}_\perp = \int_0^1 \mathbf{q}_2\mathbf{b}_\perp(x)dx = 0$

vi. $\mathbf{q}_3 \cdot \mathbf{b}_\perp = \int_0^1 \mathbf{q}_3\mathbf{b}_\perp(x)dx = 0$

vii. $(\mathbf{b} - \mathbf{b}_\perp) \cdot \mathbf{b}_\perp = \int_0^1 (\mathbf{b} - \mathbf{b}_\perp)\mathbf{b}_\perp(x)dx = 0$

viii.
$$\left(\mathbf{q}_1\cdot\mathbf{b}\right)\mathbf{q}_1(x)+\left(\mathbf{q}_2\cdot\mathbf{b}\right)\mathbf{q}_2(x)+\left(\mathbf{q}_3\cdot\mathbf{b}\right)\mathbf{q}_3(x)+\mathbf{b}_\perp(x)-\mathbf{b}(x)$$
$$=5\mathbf{q}_1(x)+3\sqrt{3}\mathbf{q}_2(x)+\sqrt{5}\mathbf{q}_3(x)+\mathbf{b}_\perp(x)-\mathbf{b}(x)$$
$$=5\cdot 1+3\sqrt{3}\cdot\sqrt{3}\ \left(2x-1\right)+\sqrt{5}\cdot\sqrt{5}\ \left(6x^2-6x+1\right)+20x^3-30x^2+12x-1-20x^3$$
$$=5+9\ \left(2x-1\right)+5\ \left(6x^2-6x+1\right)-30x^2+12x-1$$
$$=\left(5\cdot 6-30\right)x^2+\left(9\cdot 2-5\cdot 6+12\right)x+5-9+5-1=0$$

ix.
$$\text{percent relative error}-5:=100\left\|\mathbf{b}_\perp\right\|/\left\|\mathbf{b}\right\|-5=100\sqrt{(1/7)/(400/7)}-5$$
$$=100\sqrt{1/400}-5=5-5=0.$$

3. Given

$$\mathbf{v}_1=\begin{bmatrix}3\\3\\3\\3\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}3\\-5\\1\\1\end{bmatrix},\mathbf{v}_3=\begin{bmatrix}3\\1\\1\\-5\end{bmatrix},\mathbf{b}_\perp=\begin{bmatrix}3\\1\\-5\\1\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}6\\2\\-4\\-4\end{bmatrix},$$
$$\text{relative error squared}:=\left(\left\|\mathbf{b}_\perp\right\|/\left\|\mathbf{b}\right\|\right)^2,$$

compute the following quantities:

i. $\mathbf{v}_1\cdot\mathbf{v}_2$

ii. $\mathbf{v}_1\cdot\mathbf{v}_3$

iii. $\mathbf{v}_1\cdot\mathbf{b}_\perp$

iv. $\mathbf{v}_2\cdot\mathbf{v}_3$

v. $\mathbf{v}_2\cdot\mathbf{b}_\perp$

vi. $\mathbf{v}_3\cdot\mathbf{b}_\perp$

vii. $\left(\mathbf{b}-\mathbf{b}_\perp\right)\cdot\mathbf{b}_\perp$

viii. $\dfrac{\mathbf{v}_1\cdot\mathbf{b}}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1+\dfrac{\mathbf{v}_2\cdot\mathbf{b}}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2+\dfrac{\mathbf{v}_3\cdot\mathbf{b}}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3+\mathbf{b}_\perp-\mathbf{b}$

ix. relative error squared $-1/2$

20 pt.s

Solution

We have

i. $\mathbf{v}_1\cdot\mathbf{v}_2=3\cdot\left(3-5+1+1\right)=3\cdot 0=0$

ii. $\mathbf{v}_1 \cdot \mathbf{v}_3 = 3 \cdot (3+1+1-5) = 3 \cdot 0 = 0$

iii. $\mathbf{v}_1 \cdot \mathbf{b}_\perp = 3 \cdot (3+1-5+1) = 3 \cdot 0 = 0$

iv. $\mathbf{v}_2 \cdot \mathbf{v}_3 = 9-5+1-5 = 0$

v. $\mathbf{v}_2 \cdot \mathbf{b}_\perp = 9-5-5+1 = 0$

vi. $\mathbf{v}_3 \cdot \mathbf{b}_\perp = 9+1-5-5 = 0$

vii. $(\mathbf{b}-\mathbf{b}_\perp)\cdot \mathbf{b}_\perp = ((6,2,-4,-4)-(3,1,-5,1))\cdot(3,1,-5,1) = (3,1,1,-5)\cdot(3,1,-5,1) = 9+1-5-5 = 0$

viii.
$$\frac{\mathbf{v}_1 \cdot \mathbf{b}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{b}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{v}_3 \cdot \mathbf{b}}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 + \mathbf{b}_\perp - \mathbf{b} =$$
$$\frac{3\cdot(6+2-4-4)}{4\cdot 3^2}(3,3,3,3) + \frac{18-10-4-4}{9+25+1+1}(3,-5,1,1) + \frac{18+2-4+20}{9+1+1+25}(3,1,1-5)$$
$$-(3,1,1,-5)$$
$$= \frac{0}{4}(1,1,1,1) + \frac{0}{36}(3,-5,1,1) + \frac{36}{36}(3,1,1-5) - (3,1,1,-5) = (0,0,0,0)$$

ix. relative error squared $-1/2 = \left(\|\mathbf{b}_\perp\| / \|\mathbf{b}\|\right)^2 - 1/2 = (9+1+25+1)/(36+4+16+16)$
$= 1/2 - 1/2 = 0.$

4. A certain 3 dimensional subspace W of $\mathbb{R}^4$ can be expressed in the following 2 ways:

$$W = Span\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 12 \\ 6 \end{bmatrix} \right\} = Span\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{6}\begin{bmatrix} 3 \\ -5 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{6}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

Thus a vector $\mathbf{v} \in W$ can be expressed in the following 2 ways: for any such vector $\mathbf{v} \in W$ there exist unique coordinate vectors $(\mathbf{v})_S, (\mathbf{v})_{S'} \in \mathbb{R}^3$ such that

$$\mathbf{v} = \begin{bmatrix} 3 & 9 & 18 \\ 3 & 1 & 0 \\ 3 & 7 & 12 \\ 3 & 7 & 6 \end{bmatrix}(\mathbf{v})_S = \begin{bmatrix} 1/2 & 3/6 & 3/6 \\ 1/2 & -5/6 & 1/6 \\ 1/2 & 1/6 & 1/6 \\ 1/2 & 1/6 & -5/6 \end{bmatrix}(\mathbf{v})_{S'} = \frac{1}{6}\begin{bmatrix} 3 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & -5 \end{bmatrix}(\mathbf{v})_{S'}$$

(the latter representation of the "S'" matrix maybe being more convenient for you). Find the transition matrix $P_{S'S}$ from the basis $S = \{(3,3,3,3),(9,1,7,7),(18,0,12,6)\}$ to the basis

$S' = \left\{ \frac{1}{6}(3,3,3,3), \frac{1}{6}(3,-5,1,1), \frac{1}{6}(3,1,1,-5) \right\}$ by i) row-reducing $\left[S' \middle| S \right]$ to $\left[\frac{I}{\mathbf{0}} \middle| \frac{P_{S'S}}{\mathbf{0}} \right]$,
where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ S = \begin{bmatrix} 3 & 9 & 18 \\ 3 & 1 & 0 \\ 3 & 7 & 12 \\ 3 & 7 & 6 \end{bmatrix},\ S' = \frac{1}{6} \begin{bmatrix} 3 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & -5 \end{bmatrix},\ \text{and } \mathbf{0} = (0,0,0),$$

and, since it turns out that $S'^T S' = I$ (but S' is not orthogonal since S' is not square), by ii) computing

$$P_{S'S} = S'^T S. \tag{0.2}$$

[Of course if you do not get the same 3×3 matrix $P_{S'S}$ in the two approaches go back and find your error(s)! Also note that I have engineered this problem so that, despite the fractions in S' (which is a bit unavoidable if $S'^T S' = I$), $P_{S'S}$ nevertheless actually has integer entries—which suggests a "best practices" approach to row reduction—and that $P_{S'S}$ is actually upper triangular!]

Finally for

$$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} \in W,$$

iii) obtain $(\mathbf{v})_{S'} \in \mathbb{R}^3$ by inspection, and then iv) compute $(\mathbf{v})_S = P_{S'S}^{-1} (\mathbf{v})_{S'}$ by row reducing $\left[P_{S'S} \middle| (\mathbf{v})_{S'} \right]$ to $\left[I \middle| P_{S'S}^{-1} (\mathbf{v})_{S'} \right] = \left[I \middle| (\mathbf{v})_S \right]$.

[FYI: for those of you who would like to know how the second approach here generalizes when the matrices S and S' are not square, and neither one has the property that $S^T S = I$, just realize the following. If the columns of the matrices S and S' are elements of *bases*, then the columns of either matrix are independent and, according to THEOREM 6.4.3, this is equivalent to $S^T S$ and $S'^T S'$, which are square, being invertible. Thus, for example, we get the following logic:

$$\begin{aligned} & S'(\mathbf{v})_{S'} = S(\mathbf{v})_S \Rightarrow S'^T S'(\mathbf{v})_{S'} = S'^T S(\mathbf{v})_S \Rightarrow (\mathbf{v})_{S'} = \left(S'^T S'\right)^{-1} S'^T S(\mathbf{v})_S =: P_{S'S}(\mathbf{v})_S \\ & \Rightarrow P_{S'S} = \left(S'^T S'\right)^{-1} S'^T S, \end{aligned} \tag{0.3}$$

which is a very general version of (0.2), and which is reminiscent of the "least squares" development in the relevant (invertible) case. Note that if S and S' are actually square, so that their individual inverses exist, then the most general statement (0.3) reduces to the usual

$$P_{S'S} = \left(S'^T S'\right)^{-1} S'^T S = S'^{-1}\left(S'^T\right)^{-1} S'^T S = S'^{-1}S\,,$$

which is evidently of rather limited utility.]

25 pt.s

<u>Solution</u>

For part i) a "best practices" row reduction might proceed as follows:

$$\left[S' \middle| S\right] = \left[\frac{1}{6}\begin{bmatrix}3 & 3 & 3\\ 3 & -5 & 1\\ 3 & 1 & 1\\ 3 & 1 & -5\end{bmatrix}\middle|\begin{matrix}3 & 9 & 18\\ 3 & 1 & 0\\ 3 & 7 & 12\\ 3 & 7 & 6\end{matrix}\right] \sim \left[\begin{matrix}3 & 3 & 3\\ 3 & -5 & 1\\ 3 & 1 & 1\\ 3 & 1 & -5\end{matrix}\middle|\begin{matrix}18 & 54 & 108\\ 18 & 6 & 0\\ 18 & 42 & 72\\ 18 & 42 & 36\end{matrix}\right]$$

$$\sim \left[\begin{matrix}3 & 3 & 3\\ 0 & 8 & 2\\ 0 & 2 & 2\\ 0 & 2 & 8\end{matrix}\middle|\begin{matrix}18 & 54 & 108\\ 0 & 48 & 108\\ 0 & 12 & 36\\ 0 & 12 & 72\end{matrix}\right] \sim \left[\begin{matrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 4 & 1\\ 0 & 1 & 4\end{matrix}\middle|\begin{matrix}6 & 18 & 36\\ 0 & 6 & 18\\ 0 & 24 & 54\\ 0 & 6 & 36\end{matrix}\right] \sim \left[\begin{matrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 3\\ 0 & 0 & 3\end{matrix}\middle|\begin{matrix}6 & 18 & 36\\ 0 & 6 & 18\\ 0 & 0 & 18\\ 0 & 0 & 18\end{matrix}\right]$$

$$\sim \left[\begin{matrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{matrix}\middle|\begin{matrix}6 & 18 & 36\\ 0 & 6 & 18\\ 0 & 0 & 6\\ 0 & 0 & 0\end{matrix}\right] \sim \left[\begin{matrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{matrix}\middle|\begin{matrix}6 & 18 & 30\\ 0 & 6 & 12\\ 0 & 0 & 6\\ 0 & 0 & 0\end{matrix}\right] \sim \left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{matrix}\middle|\begin{matrix}6 & 12 & 18\\ 0 & 6 & 12\\ 0 & 0 & 6\\ 0 & 0 & 0\end{matrix}\right]$$

$$= \left[\begin{matrix}I\\ \hline \mathbf{0}\end{matrix}\middle|\begin{matrix}P_{S'S}\\ \hline \mathbf{0}\end{matrix}\right].$$

For ii) we get the direct computation

$$P_{S'S} = S'^T S = \frac{1}{6}\begin{bmatrix} 3 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & -5 \end{bmatrix}^T \begin{bmatrix} 3 & 9 & 18 \\ 3 & 1 & 0 \\ 3 & 7 & 12 \\ 3 & 7 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \end{bmatrix}\begin{bmatrix} 3 & 9 & 18 \\ 3 & 1 & 0 \\ 3 & 7 & 12 \\ 3 & 7 & 6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 4\cdot 3^2 & 3\cdot(9+1+7+7) & 3\cdot(18+12+6) \\ (3-5+1+1)\cdot 3 & 27-5+7+7 & 3\cdot 18+12+6 \\ (3+1+1-5)\cdot 3 & 27+1+7-35 & 3\cdot 18+12-30 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6\cdot 6 & 6\cdot 12 & 6\cdot 18 \\ 0 & 6\cdot 6 & 6\cdot 12 \\ 0 & 0 & 6\cdot 6 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 0 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix}.$$

For iii) it is clear that since

$$S'\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} = \mathbf{v}, \text{ then},$$

$$(\mathbf{v})_{S'} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}.$$

So then for iv) we get

$$\left[P_{S'S} \middle| (\mathbf{v})_{S'}\right] = \left[\begin{array}{ccc|c} 6 & 12 & 18 & 0 \\ 0 & 6 & 12 & 0 \\ 0 & 0 & 6 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left[I \middle| (\mathbf{v})_S\right].$$

5. For the following 3-dimensional subspace W of $\mathbb{R}^4$, find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ via the Gram-Schmidt procedure:

$$W = Span\left\{\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 12 \\ 6 \end{bmatrix}\right\}. \tag{0.4}$$

20 pt.s

Solution

The Gram-Schmidt procedure (suitably improved) gives the desired basis. So in (0.4) write $W = Span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to identify $\mathbf{u}_1, \mathbf{u}_2,$ and $\mathbf{u}_3$, and then compute/define (non-normalized) $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ as follows:

$$\begin{aligned}
\mathbf{u}_1 &= (3,3,3,3) \propto (1,1,1,1) =: \mathbf{v}_1 \\
\mathbf{u}_2 - proj_{Span\{\mathbf{v}_1\}}\mathbf{u}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (9,1,7,7) - \frac{(9,1,7,7)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) \\
&= (9,1,7,7) - \frac{24}{4}(1,1,1,1) = (3,-5,1,1) =: \mathbf{v}_2 \\
\mathbf{u}_3 &= (18,0,12,6) \propto (3,0,2,1) =: \mathbf{u}_3' \\
\mathbf{u}_3' - proj_{Span\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{u}_3' &= \mathbf{u}_3' - \frac{\mathbf{u}_3' \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3' \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= (3,0,2,1) - \frac{(3,0,2,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) - \frac{(3,0,2,1)\cdot(3,-5,1,1)}{(3,-5,1,1)\cdot(3,-5,1,1)}(3,-5,1,1) \\
&= (3,0,2,1) - \frac{6}{4}(1,1,1,1) - \frac{12}{36}(3,-5,1,1) \\
&= (3,0,2,1) - \frac{3}{2}(1,1,1,1) - \frac{1}{3}(3,-5,1,1) \\
&\propto 6(3,0,2,1) - 9(1,1,1,1) - 2(3,-5,1,1) = (3,1,1,-5) =: \mathbf{v}_3.
\end{aligned} \tag{0.5}$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1,1,1,1), (3,-5,1,1), (3,1,1,-5)\}$ is now an orthogonal basis, and we normalize to the desired basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ by defining

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{v}_1 / \|\mathbf{v}_1\| = \mathbf{v}_1 / (\mathbf{v}_1 \cdot \mathbf{v}_1)^{1/2} = (1,1,1,1) / \left((1,1,1,1)\cdot(1,1,1,1)\right)^{1/2} \\
&= (1,1,1,1)/(4)^{1/2} = (1,1,1,1)/2 \\
\mathbf{q}_2 &= \mathbf{v}_2 / (\mathbf{v}_2 \cdot \mathbf{v}_2)^{1/2} = (3,-5,1,1) / \left((3,-5,1,1)\cdot(3,-5,1,1)\right)^{1/2} \\
&= (3,-5,1,1)/(36)^{1/2} = (3,-5,1,1)/6 \\
\mathbf{q}_3 &= \mathbf{v}_3 / (\mathbf{v}_3 \cdot \mathbf{v}_3)^{1/2} = (3,1,1,-5) / \left((3,1,1,-5)\cdot(3,1,1,-5)\right)^{1/2} \\
&= (3,1,1,-5)/(36)^{1/2} = (3,1,1,-5)/6,
\end{aligned}$$

so that the desired basis is specified within the statement that

$$W = Span\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = Span\left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{6}\begin{bmatrix}3\\-5\\1\\1\end{bmatrix}, \frac{1}{6}\begin{bmatrix}3\\1\\1\\-5\end{bmatrix}\right\}.$$

6. Perform the QR factorization of the matrix

$$A = \begin{bmatrix}3 & 9 & 18\\3 & 1 & 0\\3 & 7 & 12\\3 & 7 & 6\end{bmatrix},$$

i.e. write $A = QR$ with Q having (three) orthonormal column vectors (each in $\mathbb{R}^4$) and R being a 3×3 upper triangular matrix (which will be invertible here, as per the general theory). Secondly, compute i) $Q^T\mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix}6\\2\\-4\\-4\end{bmatrix},$$

and then compute ii) $QQ^T\mathbf{b}$, iii) $QQ^T\mathbf{b}-\mathbf{b}$, as well as iv) the inner product of $QQ^T\mathbf{b}$ with $QQ^T\mathbf{b}-\mathbf{b}$. (If you don't get zero in the last instance, try again! I will give you points for showing that it is zero, or a bit less for noticing that you committed an error somewhere if it isn't. Also note that $QQ^T \neq I$ since Q is not square!)

15 pt.s

<u>Solution</u>

The QR factorization of a matrix A with independent columns is effected by performing Gram-Schmidt on the columns of A, which, together with normalization, gives the columns of Q. And then, since $Q^TQ = I$ (since Q's columns are orthonormal), R is obtained by computing Q^TA. For the matrix at hand, Q was computed in the previous problem. There we found

$$Q = \left[\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \middle| \frac{1}{6}\begin{bmatrix}3\\-5\\1\\1\end{bmatrix} \middle| \frac{1}{6}\begin{bmatrix}3\\1\\1\\-5\end{bmatrix} \right],$$

where we reduce the writing of fractions by a certain (hopefully) obvious notation. So then, as per the general theory just discussed,

$$R = Q^T A = \begin{bmatrix}(1,1,1,1)/2\\(3,-5,1,1)/6\\(3,1,1-5)/6\end{bmatrix}\begin{bmatrix}3&9&18\\3&1&0\\3&7&12\\3&7&6\end{bmatrix} = \begin{bmatrix}(3+3+3+3)/2 & (9+1+7+7)/2 & (18+12+6)/2\\(9-15+3+3)/6 & (27-5+7+7)/6 & (54+12+6)/6\\(9+3+3-15)/6 & (27+1+7-35)/6 & (54+12-30)/6\end{bmatrix}$$

$$= \begin{bmatrix}6&12&18\\0&6&12\\0&0&6\end{bmatrix}.$$

Check: Note that

$$QR = \left[\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \middle| \frac{1}{6}\begin{bmatrix}3\\-5\\1\\1\end{bmatrix} \middle| \frac{1}{6}\begin{bmatrix}3\\1\\1\\-5\end{bmatrix} \right]\begin{bmatrix}6&12&18\\0&6&12\\0&0&6\end{bmatrix} = \left[\begin{bmatrix}3\\3\\3\\3\end{bmatrix} \middle| \begin{bmatrix}3\\-5\\1\\1\end{bmatrix} \middle| \begin{bmatrix}3\\1\\1\\-5\end{bmatrix} \right]\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$$

$$= \left[1\begin{bmatrix}3\\3\\3\\3\end{bmatrix} \middle| 2\begin{bmatrix}3\\3\\3\\3\end{bmatrix}+1\begin{bmatrix}3\\-5\\1\\1\end{bmatrix} \middle| 3\begin{bmatrix}3\\3\\3\\3\end{bmatrix}+2\begin{bmatrix}3\\-5\\1\\1\end{bmatrix}+1\begin{bmatrix}3\\1\\1\\-5\end{bmatrix} \right] = \left[\begin{bmatrix}3\\3\\3\\3\end{bmatrix} \middle| \begin{bmatrix}9\\1\\7\\7\end{bmatrix} \middle| \begin{bmatrix}18\\0\\12\\6\end{bmatrix} \right] = A$$

Now we compute $Q^T\mathbf{b}$,

$$Q^T\mathbf{b} = \begin{bmatrix}(1,1,1,1)/2\\(3,-5,1,1)/6\\(3,1,1-5)/6\end{bmatrix}\begin{bmatrix}6\\2\\-4\\-4\end{bmatrix} = \begin{bmatrix}3+1-2-2\\(9-5-2-2)/3\\(3\cdot 3+1-2+10)/3\end{bmatrix} = \begin{bmatrix}0\\0\\6\end{bmatrix} = 6\begin{bmatrix}0\\0\\1\end{bmatrix},$$

and then $QQ^T\mathbf{b}$,

$$QQ^T\mathbf{b} = \left[\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \middle| \frac{1}{6}\begin{bmatrix}3\\-5\\1\\1\end{bmatrix} \middle| \frac{1}{6}\begin{bmatrix}3\\1\\1\\-5\end{bmatrix} \right] 6\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\1\\1\\-5\end{bmatrix}$$

as well as $QQ^T\mathbf{b}-\mathbf{b}$,

$$QQ^T\mathbf{b}-\mathbf{b} = \begin{bmatrix}3\\1\\1\\-5\end{bmatrix} - \begin{bmatrix}6\\2\\-4\\-4\end{bmatrix} = \begin{bmatrix}-3\\-1\\5\\-1\end{bmatrix} = -\begin{bmatrix}3\\1\\-5\\1\end{bmatrix},$$

and finally $\left(QQ^T\mathbf{b}-\mathbf{b}\right)\cdot QQ^T\mathbf{b}$,

$$\left(QQ^T\mathbf{b}-\mathbf{b}\right)\cdot QQ^T\mathbf{b} = -\begin{bmatrix}3\\1\\-5\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\\-5\end{bmatrix} = -\left(9+1-5-5\right) = 0.$$

7. Find the least squares solution $\mathbf{x}$ of the system $A\mathbf{x}=\mathbf{b}$, where

$$A = \begin{bmatrix}3 & 9 & 18\\3 & 1 & 0\\3 & 7 & 12\\3 & 7 & 6\end{bmatrix}, \ \mathbf{b} = \begin{bmatrix}6\\2\\-4\\-4\end{bmatrix}.$$

With your least squares solution $\mathbf{x}$, compute i) $A\mathbf{x}$, ii) $A\mathbf{x}-\mathbf{b}$ and then iii) the inner product of $A\mathbf{x}$ with $A\mathbf{x}-\mathbf{b}$. (If you don't get zero in the last instance, try again! I will give you points for showing that it is zero, or a bit less for noticing that you committed an error somewhere if it isn't.)

15 pt.s

<u>Solution</u>

The least squares solution $\mathbf{x}$ of the system $A\mathbf{x}=\mathbf{b}$ is the ("real") solution of the normal system $A^TA\mathbf{x}=A^T\mathbf{b}$. This is most easily accomplished with the QR factorization already in hand by noting that if $A=QR$, with $Q^TQ=I$, then $A^TA=\left(QR\right)^T QR = R^TQ^TQR = R^TR$

and $A^T\mathbf{b} = (QR)^T\,\mathbf{b} = R^T Q^T\mathbf{b}$, so that the normal system becomes $R^T R\mathbf{x} = R^T Q^T\mathbf{b}$. But then the latter equation, since R (and then R^T) is invertible, gives $R\mathbf{x} = Q^T\mathbf{b}$. For the present A specified, and as per the results of the previous problem, that system is

$$\begin{bmatrix} 6 & 12 & 18 \\ 0 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix}\mathbf{x} = R\mathbf{x} = Q^T\mathbf{b} = 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

for which row reduction (or back substitution, since R is upper triangular) gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left[I \middle| \mathbf{x}\right],$$

$\mathbf{x}$ the desired least squares solution.

Continuing with the other requested components one finds that

$$A\mathbf{x} = \begin{bmatrix} 3 & 9 & 18 \\ 3 & 1 & 0 \\ 3 & 7 & 12 \\ 3 & 7 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix},$$

and then that

$$A\mathbf{x} - \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 5 \\ -1 \end{bmatrix} = -\begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

and finally that

$$(A\mathbf{x} - \mathbf{b}) \cdot A\mathbf{x} = -\begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} = -(9 + 1 - 5 - 5) = 0.$$

8. Find the orthogonal projection $proj_W \mathbf{b}$ of $\mathbf{b} = (6,2,-4,-4)$ onto the following subspace of $\mathbb{R}^4$:

$$W = Span\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 12 \\ 6 \end{bmatrix} \right\}.$$

Check your answer by making sure that $\mathbf{b} - proj_W \mathbf{b}$ is orthogonal to $proj_W \mathbf{b}$ (I'm asking you to compute the inner product of your answer $proj_W \mathbf{b}$ with $\mathbf{b} - proj_W \mathbf{b}$, which better be zero! I will give you points for showing that it is, or a bit less for noticing that you committed an error somewhere if it isn't.)

10 pt.s

Solution

As per the usual theory, $proj_W \mathbf{b} = A\mathbf{x}$ where $\mathbf{x}$ is the least squares solution of the system $A\mathbf{x} = \mathbf{b}$. So from the previous problem we have

$$proj_W \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 3 & 9 & 18 \\ 3 & 1 & 0 \\ 3 & 7 & 12 \\ 3 & 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix},$$

and then

$$\mathbf{b} - proj_W \mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ -4 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix},$$

and finally

$$(\mathbf{b} - proj_W \mathbf{b}) \cdot proj_W \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix} = 9 + 1 - 5 - 5 = 0$$

as before.

9. Find the orthogonal complement $W^\perp$ of

$$W = Span\left\{\begin{bmatrix}3\\3\\3\\3\end{bmatrix},\begin{bmatrix}9\\1\\7\\7\end{bmatrix},\begin{bmatrix}18\\0\\12\\6\end{bmatrix}\right\}$$

(i.e. compute a basis for such) and, secondly, check that $\mathbf{b} - proj_W\mathbf{b}$ from the previous problem is in this space. (I will give you points for showing that $\mathbf{b} - proj_W\mathbf{b} \in W^\perp$, or a bit less for noticing that you committed an error somewhere if it isn't.)

10 pt.s

<u>Solution:</u>

$W^\perp$ is the null space of the matrix B whose rows are the basis elements of W (because, by theorem, the null space of a matrix B is the orthogonal complement of the row space of B—and vice versa). So we have

$$W^\perp = \text{Null}\begin{bmatrix}3&3&3&3\\9&1&7&7\\18&0&12&6\end{bmatrix} = \text{Null}\begin{bmatrix}1&1&1&1\\9&1&7&7\\3&0&2&1\end{bmatrix} = \text{Null}\begin{bmatrix}1&1&1&1\\0&-8&-2&-2\\0&-3&-1&-2\end{bmatrix}$$

$$= \text{Null}\begin{bmatrix}1&1&1&1\\0&4&1&1\\0&3&1&2\end{bmatrix} = \text{Null}\begin{bmatrix}1&1&1&1\\0&1&0&-1\\0&3&1&2\end{bmatrix} = \text{Null}\begin{bmatrix}1&1&1&1\\0&1&0&-1\\0&0&1&5\end{bmatrix}$$

$$= \text{Null}\begin{bmatrix}1&1&0&-4\\0&1&0&-1\\0&0&1&5\end{bmatrix} = \text{Null}\begin{bmatrix}1&0&0&-3\\0&1&0&-1\\0&0&1&5\end{bmatrix} = Span\left\{\begin{bmatrix}3\\1\\-5\\1\end{bmatrix}\right\},$$

which subspace clearly contains

$$\mathbf{b} - proj_W\mathbf{b} = \begin{bmatrix}6\\2\\-4\\-4\end{bmatrix} - \begin{bmatrix}3\\1\\1\\-5\end{bmatrix} = \begin{bmatrix}3\\1\\-5\\1\end{bmatrix}.$$

10. Prove that for a square matrix $A = A_{n\times n}$, if

$$A\mathbf{x}\cdot A\mathbf{y}=\mathbf{x}\cdot\mathbf{y} \tag{0.6}$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, then $A^TA=I=I_{n\times n}$, so that A must be orthogonal.

10 pt.s

Solution

(0.6) implies that

$$\mathbf{x}\cdot\mathbf{y}=A\mathbf{x}\cdot A\mathbf{y}=\mathbf{x}\cdot A^TA\mathbf{y}\Leftrightarrow 0=\mathbf{x}\cdot\left(A^TA\mathbf{y}-\mathbf{y}\right)=\mathbf{x}\cdot\left(A^TA-I\right)\mathbf{y}\ . \tag{0.7}$$

Since (0.7) holds for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, it holds for $\mathbf{x}=\left(A^TA-I\right)\mathbf{y}$ so that (0.7) becomes

$$0=\left(A^TA-I\right)\mathbf{y}\cdot\left(A^TA-I\right)\mathbf{y}\Leftrightarrow\left(A^TA-I\right)\mathbf{y}=\mathbf{0}. \tag{0.8}$$

Since (0.8) holds for all $\mathbf{y}\in\mathbb{R}^n$, the null space of A^TA-I is $\mathbb{R}^n$, giving the nullity of A^TA-I is n and A^TA-I's rank is 0 (since, theorem, rank M + nullity M = number of columns of M). But the rank of a matrix is the number of its independent column (or row) vectors. Since it has none, each column must be the zero vector, giving $A^TA-I=\left[0\right]\Leftrightarrow A^TA=I$, as claimed.